TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 4, April 1997, Pages 1685–1696
S 0002-9947(97)01841-2

NONEXISTENCE OF GLOBAL SOLUTIONS OF A NONLINEAR HYPERBOLIC SYSTEM

KENG DENG

Abstract. Consider the initial value problem

$$\begin{aligned} &u_{tt} = \Delta u + |v|^p, \quad v_{tt} = \Delta v + |u|^q, \quad x \in \mathbb{R}^n, \quad t > 0,\\ &u(x,0) = f(x), \qquad v(x,0) = h(x),\\ &u_t(x,0) = g(x), \quad\ \ v_t(x,0) = k(x), \end{aligned}$$

with $1 \le n \le 3$ and $p, q > 0$. We show that there exists a bound $B(n)$ $(\le \infty)$ such that if $1 < pq < B(n)$ all nontrivial solutions with compact support blow up in finite time.

1. Introduction

This paper is concerned with the initial value problem for a nonlinear hyperbolic system as follows:

$$\begin{aligned} &u_{tt} = \Delta u + |v|^p, \quad v_{tt} = \Delta v + |u|^q, \quad x \in \mathbb{R}^n, \quad t > 0,\\ &u(x,0) = f(x), \qquad v(x,0) = h(x),\\ &u_t(x,0) = g(x), \quad\ \ v_t(x,0) = k(x). \end{aligned} \tag{1.1}$$

Here $1 \le n \le 3$, $p, q > 0$ and $pq > 1$, and the initial values are compactly supported. Such a system is a special case of a significant class of quasilinear second order hyperbolic systems with application in physics and applied science. For details, we refer the reader to [5] and the literature cited therein.

In order to motivate the main results for system (1.1), we recall an old result for the initial value problem of a single equation

$$\begin{aligned} &u_{tt} = \Delta u + |u|^p, \qquad x \in \mathbb{R}^n, \quad t > 0,\\ &u(x,0) = f(x), \quad u_t(x,0) = g(x). \end{aligned} \tag{1.2}$$

Theorem 0. (a) *For $n = 1$, if $1 < p < \infty$, every nontrivial solution of (1.2) blows up in finite time.*

(b) *For $n = 2, 3$, there exists a critical exponent $p_0(n)$ such that if $1 < p \le p_0(n)$ (1.2) has no nontrivial global solutions, while it admits nontrivial global small solutions if $p > p_0(n)$, where $p_0(2) = (3 + \sqrt{17})/2$ and $p_0(3) = 1 + \sqrt{2}$.*

Statement (a) was established by Glassey, Kato, and Sideris [3], [7], [12]. In statement (b) the subcritical case was proved by John when $n = 3$ [6] and by Glassey for $n = 2, 3$ [3]; the critical case was proved by Schaeffer for $n = 2, 3$ [11];

Received by the editors March 16, 1995 and, in revised form, December 1, 1995.
1991 *Mathematics Subject Classification.* Primary 35L15, 35L55, 35L70.

and the supercritical case was proved by Glassey when $n = 2$ [4] and by John for $n = 3$ [6].

Such a result is the exact analogue of Fujita's theorem [2] for the problem of a nonlinear parabolic equation

$$\begin{aligned} &u_t = \Delta u + |u|^p, \quad x \in \mathbb{R}^n, \quad t > 0, \\ &u(x,0) = f(x) \end{aligned} \tag{1.3}$$

with nonnegative initial data f. He showed that (i) if $1 < p < 1 + 2/n$, then (1.3) possesses no global nonnegative solutions while (ii) if $p > 1 + 2/n$, both global and nonglobal nonnegative solutions exist. Weissler [14] proved that the critical value belongs to case (i).

Escobedo and Herrero [1] extended Fujita's result to the initial value problem for a weakly coupled system

$$\begin{aligned} &u_t = \Delta u + |v|^p, \quad v_t = \Delta v + |u|^q, \quad x \in \mathbb{R}^n, \quad t > 0, \\ &u(x,0) = f(x), \quad v(x,0) = h(x). \end{aligned} \tag{1.4}$$

They proved that if $1 < pq \le 1 + 2(\max(p,q) + 1)/n$ then there exist no global nonnegative solutions of (1.4), but if $pq > 1 + 2(\max(p,q) + 1)/n$ then there are both global and nonglobal nonnegative solutions. Meanwhile, Levine [10] extended Fujita's result, in the Lipschitz case, to the system of the same equations as in (1.4) with $\mathbb{R}^n$ replaced by a cone or by the exterior of a bounded domain.

From the above discussion, a natural question arises: Can one extend the result for problem (1.2) to system (1.1)? Some efforts were made in that direction. On the one hand, based on the method of invariant norms, Klainerman [8] established the global existence result for the solution of system (1.1) in $\mathbb{R}^3$ with $p = q = 3$ and small initial data. On the other hand, via the concavity method, Levine [9] showed that for system (1.1) with $p, q > 1$ and sufficiently large initial data the solution blows up in finite time. In this paper, we also intend to give a generalization of the blow–up result in Theorem 0 to system (1.1). Specifically, we shall prove that there exists a bound $B(n)$ ($\le \infty$) such that if $1 < pq < B(n)$ all nontrivial solutions of (1.1) blow up in finite time. To achieve our objective, however, there are at least two technical difficulties to overcome. First, previous arguments used to prove Theorem 0 can only apply to convex nonlinearities such as $|u|^p$ for $p > 1$, whereas we shall also consider the concave case $0 < p < 1$. Secondly, even for the case $p, q > 1$, if we follow the general ideas such as in [3], [7], [13], we will obtain a pair of ordinary differential inequalities, which appear not amenable to analysis. Therefore in the sequel we shall employ a quite different approach, which strongly relies on the positivity of the fundamental solution of the wave equation, and thus we are restricted to dimensions $1 \le n \le 3$.

2. Global nonexistence theorems

In this section we present the main results concerning the nonexistence of global solutions of system (1.1). For that purpose, we introduce the Riemann function $R(t)$ for the wave equation. It is a positive operator in $\mathbb{R}^n$ ($1 \le n \le 3$). From the integral representation formula, the classical solution (u, v) of (1.1) satisfies

$$u(t) = u_0(t) + \int_0^t R(t-\tau) * |v(\tau)|^p d\tau, \tag{2.1a}$$

$$v(t) = v_0(t) + \int_0^t R(t-\tau) * |u(\tau)|^q d\tau, \tag{2.1b}$$

where u_0 and v_0 are solutions of $w_{tt} = \Delta w$ with the same initial data as u and v, respectively.

In what follows, we only need (u, v) to be a weak solution of (1.1) in the sense that on a time interval $0 \le t < T < \infty$ $u(t) \in C([0,T]; L^q(\mathbb{R}^n))$, $v(t) \in C([0,T]; L^p(\mathbb{R}^n))$, and u, v satisfy the integral equations (2.1a)–(2.1b). Since our main interest lies in the global nonexistence results, we shall not specify conditions on the initial data. Instead we impose several assumptions on (u, v) and (u_0, v_0). For definiteness, we may assume $p \le q$ throughout this section.

(H)

i) (u, v) *are compactly supported with* $\operatorname{supp}\{u, v\} \subseteq \{|x| < t + d\}$;

ii) *For* $0 < p < 1$, $\int_{\mathbb{R}^n} u_0(x,t)dx \ge 0$, $v_0(x,t) \ge 0$, *and* $\|v_0(t)\|_p \ge \tilde{c}_k t$;

iii) *For* $p \ge 1$, $\int_{\mathbb{R}^n} u_0(x,t)dx \ge 0$ *and* $\int_{\mathbb{R}^n} v_0(x,t)dx \ge c_k t$,

where $\tilde{c}_k$ and c_k are positive constants with $c_k \equiv \int_{\mathbb{R}^n} k(x)dx$.

Remark 1. The conditions in (H) are fairly reasonable, since for the classical solution of $w_{tt} = \Delta w$ with compactly supported initial data $w(x, 0) = f$ and $w_t(x, 0) = g$, $\left(d^2/dt^2\right) \int_{\mathbb{R}^n} w(x,t)dx = 0$. Hence $\int_{\mathbb{R}^n} w(x,t)dx = c_g t + c_f$, where $c_g = \int g dx$ and $c_f = \int f dx$.

We first establish the following two results.

Lemma 1. *Let ψ be a nonnegative solution of the integral equation*

$$\psi(t) = c_1 t^\alpha (t+d)^{-\gamma} + c_2 (t+d)^{-\sigma} \int_0^t (t-\tau)^3 \left(\frac{\tau}{\tau+d}\right)^\beta \psi^\mu(\tau) d\tau, \tag{2.2}$$

where c_1 and c_2 are positive constants. If $\alpha \ge \gamma \ge 0$, $\beta \ge 0$, $0 < \sigma < 4$, and $\mu > 1$, then $\psi(t)$ can only exist on $[0, T_0)$ for some $T_0 < \infty$.

Proof. Assume to the contrary that $\psi(t)$ exists globally on $[0, \infty)$. Then for any positive number T $(T \ge 2d)$, we find

$$\psi(t) \ge c_1 2^{-\gamma} d^{\alpha-\gamma} + c_2 2^{-(\beta+\sigma)} T^{-\sigma} \int_d^t (t-\tau)^3 \psi^\mu(\tau) d\tau \quad \text{for } d \le t \le T.$$

Thus by comparison, $\psi(t) \ge \varphi(t)$ on $[d, T]$, where

$$\varphi(t) = \delta + c_3 T^{-\sigma} \int_d^t (t-\tau)^3 \varphi^\mu(\tau) d\tau \quad \text{for } d \le t \le T \tag{2.3}$$

with $\delta = c_1 2^{-\gamma} d^{\alpha-\gamma}$ and $c_3 = c_2 2^{-(\beta+\sigma)}$. Clearly, $\varphi(t)$ satisfies

$$\begin{aligned} &\varphi^{(iv)}(t) = 6c_3 T^{-\sigma} \varphi^\mu(t), \qquad d < t < T, \\ &\varphi(d) = \delta, \quad \varphi'(d) = \varphi''(d) = \varphi'''(d) = 0. \end{aligned} \tag{2.4}$$

For $0 < \sigma < 4$, if T is large enough, there exists a $\hat{T}$ ($d < \hat{T} \le T/2$) such that $\varphi(\hat{T}) = 2\delta$. Moreover,

$$\begin{aligned} \varphi''(t) &= 6c_3 T^{-\sigma} \int_d^t (t-\tau)\varphi^\mu(\tau)d\tau \\ &\ge 6c_3 T^{-\sigma} t^{-1} \int_d^t (t-\tau)^2 \varphi^\mu(\tau)d\tau \ge 2T^{-1}\varphi'(t) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \varphi'''(t) &= 6c_3 T^{-\sigma} \int_d^t \varphi^\mu(\tau)d\tau \\ &\ge 6c_3 T^{-\sigma} t^{-2} \int_d^t (t-\tau)^2 \varphi^\mu(\tau)d\tau \ge 2T^{-2}\varphi'(t). \end{aligned} \tag{2.6}$$

Combining (2.4) and (2.6) then yields

$$\varphi^{(\mathrm{iv})}(t)\varphi'''(t) \ge 12c_3 T^{-(\sigma+2)}\varphi^\mu(t)\varphi'(t),$$

which upon integration from d to t, leads to

$$\varphi'''(t) \ge cT^{-\frac{\sigma+2}{2}} \left(\varphi^{\mu+1}(t) - \delta^{\mu+1}\right)^{\frac{1}{2}} \tag{2.7}$$

with $c = [24c_3/(\mu+1)]^{1/2}$. From now on, without causing any confusion, we may use c to denote various positive constants. By means of (2.5) we further obtain

$$\varphi'''(t)\varphi''(t) \ge cT^{-\frac{\sigma+4}{2}} \left(\varphi^{\mu+1}(t) - \delta^{\mu+1}\right)^{\frac{1}{2}} \varphi'(t). \tag{2.8}$$

Integrating (2.8) over (d,t) for $\hat{T} \le t < T$ then gives

$$\begin{aligned} \left[\varphi''(t)\right]^2 &\ge cT^{-\frac{\sigma+4}{2}} \int_d^t \left(\varphi^{\mu+1}(\tau) - \delta^{\mu+1}\right)^{\frac{1}{2}} \varphi'(\tau)d\tau \\ &\ge cT^{-\frac{\sigma+4}{2}} \left[(\mu+1)^{\frac{1}{2}}\delta^{\frac{\mu}{2}} \int_\delta^{2\delta} (z-\delta)^{\frac{1}{2}}dz + c_4 \int_{2\delta}^{\varphi(t)} z^{\frac{\mu+1}{2}}dz\right], \end{aligned}$$

where $c_4 = \min\left\{(\mu+3)(\mu+1)^{1/2}2^{-(\mu+3)/2}/3, 2^{-(\mu+1)/2}\right\}$. Hence for $\hat{T} \le t < T$

$$\varphi''(t) \ge cT^{-\frac{\sigma+4}{4}}\varphi^{\frac{\mu+3}{4}}(t),$$

or

$$\varphi''(t)\varphi'(t) \ge cT^{-\frac{\sigma+4}{4}}\varphi^{\frac{\mu+3}{4}}(t)\varphi'(t).$$

The above inequality implies that for $T/2 \le t < T$

$$\begin{aligned} \varphi'(t) &\ge cT^{-\frac{\sigma+4}{8}} \left(\varphi^{\frac{\mu+7}{4}}(t) - \varphi^{\frac{\mu+7}{4}}(\hat{T})\right)^{\frac{1}{2}} \\ &\ge cT^{-\frac{\sigma+4}{8}} \left(\varphi^{\frac{\mu+7}{4}}(t) - \varphi^{\frac{\mu+7}{4}}(T/2)\right)^{\frac{1}{2}}. \end{aligned} \tag{2.9}$$

Integration of this relation over $(T/2, T)$ then leads to

$$
\begin{aligned}
cT^{\frac{4-\sigma}{8}} &\le \int_{\varphi(T/2)}^{\varphi(T)} \left(z^{\frac{\mu+7}{4}} - \varphi^{\frac{\mu+7}{4}}(T/2) \right)^{-\frac{1}{2}} dz \\
&\le 2(\mu+7)^{-\frac{1}{2}} \varphi^{-\frac{\mu+3}{8}}(T/2) \int_{\varphi(T/2)}^{2\varphi(T/2)} (z - \varphi(T/2))^{-\frac{1}{2}} dz \\
&\quad + 2^{\frac{\mu+7}{8}} \int_{2\varphi(T/2)}^{\infty} z^{-\frac{\mu+7}{8}} dz \\
&= \left[4(\mu+7)^{-\frac{1}{2}} + 16(\mu-1)^{-1} \right] \varphi^{-\frac{\mu-1}{8}}(T/2) \\
&\le \left[4(\mu+7)^{-\frac{1}{2}} + 16(\mu-1)^{-1} \right] \delta^{-\frac{\mu-1}{8}}.
\end{aligned}
\tag{2.10}
$$

Since $c = c(\beta, \sigma, \mu)$ and $\delta = \delta(d, \alpha, \gamma)$, if T is sufficiently large, (2.10) yields a contradiction, which means that $\psi(t)$ cannot exist globally.

Lemma 2. *For the integral equation*

$$
\begin{aligned}
\psi(t) &= c_1 t^{\alpha} (t+d)^{-\gamma} \\
&\quad + c_2 (t+d)^{-\sigma} \ln^{-\kappa}(1+t+d) \int_0^t (t-\tau)^3 \left(\frac{\tau}{\tau+d} \right)^{\beta} \psi^{\mu}(\tau) d\tau,
\end{aligned}
\tag{2.11}
$$

if $\beta, \kappa \ge 0,\ 0 < \sigma < 4,\ \mu > 1,$ *and* $0 < \gamma - \alpha < (4-\sigma)/(\mu-1)$, *then there exist no global nonnegative solutions.*

Proof. We will derive a contradiction by assuming that $\psi(t)$ is a global solution. Let $\psi(t) \ge \phi(t)$ on $[d, T]$, where

$$
\phi(t) = c_1 2^{-\gamma} T^{-(\gamma-\alpha)} + c_2 2^{-(\beta+\sigma)} T^{-\sigma} \ln^{-\kappa}(1+2T) \int_d^t (t-\tau)^3 \phi^{\mu}(\tau) d\tau. \tag{2.12}
$$

Proceeding essentially the same as in the proof of Lemma 1 with every δ replaced by $c_1 2^{-\gamma} T^{-(\gamma-\alpha)}$, we finally obtain

$$
\begin{aligned}
cT^{\frac{4-\sigma}{8}} \ln^{-\frac{\kappa}{8}}(1+2T) &\le \left[4(\mu+7)^{-\frac{1}{2}} + 16(\mu-1)^{-1} \right] \\
&\quad \cdot (c_1 2^{-\gamma})^{-\frac{\mu-1}{8}} T^{\frac{(\gamma-\alpha)(\mu-1)}{8}},
\end{aligned}
\tag{2.13}
$$

which is impossible for T large enough.

We then need the following three estimates.

Lemma 3. *Assume the hypothesis* (ii) *in* (H). *If* $0 < p < 1$, *then there exist two positive constants* $\tilde{c}_1$ *and* $\tilde{c}_2$ *such that the weak solution component* u *of (1.1) satisfies*

$$
\|u(t)\|_1 \ge \tilde{c}_1 t^{2+p} + \tilde{c}_2 (t+d)^{-2(1-p)} \int_0^t (t-\tau)^3 \|u(\tau)\|_{pq}^{pq} d\tau. \tag{2.14}
$$

Proof. For simplicity we let $\mu = pq$. We first estimate $\|v(t)\|_p^p$. By (2.1b) and the inequality

$$
\| |r| + |s| \|_p^p \ge (\|r\|_p + \|s\|_p)^p \ge 2^{p-1} \left(\|r\|_p^p + \|s\|_p^p \right),
$$

one can see that

$$\|v(t)\|_p^p \geq \left\{ \|v_0(t)\|_p + \left[\int_{\mathbb{R}^n} \left(\int_0^t R(t-\tau) * |u(\tau)|^q d\tau \right)^p dx \right]^{\frac{1}{p}} \right\}^p$$
$$\geq 2^{p-1} \left[\|v_0(t)\|_p^p + \int_{\mathbb{R}^n} \left(\int_0^t R(t-\tau) * |u(\tau)|^q d\tau \right)^p dx \right].$$

Applying the inverse Hölder inequality then yields

$$\begin{aligned} \|v(t)\|_p^p &\geq 2^{p-1} \tilde{c}_k^p t^p + 2^{p-1} \int_{\mathbb{R}^n} \int_0^t t^{p-1} (t-\tau)^{p-1} R(t-\tau) * |u(\tau)|^\mu d\tau dx \\ &\geq 2^{p-1} \tilde{c}_k^p t^p + 2^{p-1} (t+d)^{-2(1-p)} \int_{\mathbb{R}^n} \int_0^t R(t-\tau) * |u(\tau)|^\mu d\tau dx \\ &= 2^{p-1} \tilde{c}_k^p t^p + 2^{p-1} (t+d)^{-2(1-p)} \int_0^t (t-\tau) \|u(\tau)\|_\mu^\mu d\tau. \end{aligned} \tag{2.15}$$

We next estimate $\|u(t)\|_1$. By (2.1a) and (2.15), we find that

$$\begin{aligned} \|u(t)\|_1 &\geq \int_{\mathbb{R}^n} u(x,t) dx \\ &\geq \int_{\mathbb{R}^n} \int_0^t R(t-\eta) * |v(\eta)|^p d\eta dx \\ &= \int_0^t (t-\eta) \|v(\eta)\|_p^p d\eta \\ &\geq 2^{p-1} \tilde{c}_k^p B(2, 1+p) t^{2+p} \\ &\quad + 2^{p-1} \int_0^t (\eta+d)^{-2(1-p)} (t-\eta) \int_0^\eta (\eta-\tau) \|u(\tau)\|_\mu^\mu d\tau d\eta \\ &\geq 2^{p-1} \tilde{c}_k^p B(2, 1+p) t^{2+p} \\ &\quad + 2^{p-1} (t+d)^{-2(1-p)} \int_0^t \|u(\tau)\|_\mu^\mu \int_\tau^t (t-\eta)(\eta-\tau) d\eta d\tau \\ &= 2^{p-1} \tilde{c}_k^p B(2, 1+p) t^{2+p} + 2^{p-1} B(2,2) (t+d)^{-2(1-p)} \int_0^t (t-\tau)^3 \|u(\tau)\|_\mu^\mu d\tau, \end{aligned}$$

where $B(2, 1+p)$ and $B(2,2)$ are the Beta function. Thus we obtain the estimate (2.14).

Remark 2. It should be pointed out that the validity of (2.15) is based on the formula:

For any continuous function $\zeta(\cdot, t) \in L^\mu(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \int_0^t R(t-\tau) * |\zeta(\tau)|^\mu d\tau dx = \int_0^t (t-\tau) \|\zeta(\tau)\|_\mu^\mu d\tau,$$

which holds only for $1 \leq n \leq 3$ due to the representation form of R. Therefore the argument used herein appears not readily to extend to higher dimensions.

Lemma 4. *Assume the hypotheses* (i) *and* (iii) *in* (H). *If* $1 \leq p \leq q$, *then there exist two positive constants* $\hat{c}_1$ *and* $\hat{c}_2$ *such that the weak solution component* v *of*

(1.1) satisfies

$$\|v(t)\|_1 \geq \hat{c}_1 t + \hat{c}_2 (t+d)^{-(n+2)(q-1)} \left[\int_0^t (t-\tau)^3 \|v(\tau)\|_p^p d\tau \right]^q \tag{2.16}$$

for $n = 1, 2$.

Proof. Since u is compactly supported, by Hölder's inequality we get

$$\|u(t)\|_1 = \int\limits_{|x|<t+d} |u| dx \leq c \|u(t)\|_q (t+d)^{\frac{n(q-1)}{q}} \tag{2.17}$$

for $n = 1, 2$. By (2.1a) and (2.17) we then find

$$\begin{aligned} \|u(t)\|_q^q &\geq c^{-q} \|u(t)\|_1^q (t+d)^{-n(q-1)} \\ &\geq c^{-q} (t+d)^{-n(q-1)} \left[\int_0^t (t-\tau) \|v(\tau)\|_p^p d\tau \right]^q . \end{aligned} \tag{2.18}$$

Upon application of Jensen's inequality, there follows

$$\begin{aligned} \|v(t)\|_1 &\geq \int_{\mathbb{R}^n} v_0(x,t) dx + \int_0^t (t-\eta) \|u(\eta)\|_q^q d\eta \\ &\geq c_k t + c^{-q} \int_0^t (\eta+d)^{-n(q-1)} (t-\eta) \left[\int_0^\eta (\eta-\tau) \|v(\tau)\|_p^p d\tau \right]^q d\eta \\ &\geq c_k t + c^{-q} 2^{q-1} (t+d)^{-(n+2)(q-1)} \left[\int_0^t (t-\eta) \int_0^\eta (\eta-\tau) \|v(\tau)\|_p^p d\tau d\eta \right]^q \\ &= c_k t + c^{-q} 2^{q-1} B^q(2,2) (t+d)^{-(n+2)(q-1)} \left[\int_0^t (t-\tau)^3 \|v(\tau)\|_p^p d\tau \right]^q , \end{aligned}$$

which completes the derivation of (2.16).

Lemma 5. *Assume the hypotheses* (i) *and* (iii) *in* (H). *If* $p > 2$ *when* $n = 2$ *and* $p > 1$ *when* $n = 3$, *then there exists a positive constant* c_0 *such that the weak solution component* u *of (1.1) satisfies*

$$\|u(t)\|_1 \geq c_0 t^{2+p} (t+d)^{-\lambda} \ln^{-\nu}(1+t+d), \tag{2.19}$$

where

$$\lambda = \begin{cases} (3p-2)/2 & \text{if } n = 2, \\ 2(p-1) & \text{if } n = 3 \end{cases} \quad \text{and} \quad \nu = \begin{cases} (p-2)/2 & \text{if } n = 2, \\ 0 & \text{if } n = 3. \end{cases}$$

Proof. Arguing in a similar manner as that in the proof of Lemma 1 of [3], one can show

$$\|v(t)\|_p^p \geq c t^p (t+d)^{-\lambda} \ln^{-\nu}(1+t+d). \tag{2.20}$$

Thus we find that

$$\begin{aligned} \|u(t)\|_1 &\geq \int_0^t (t-\eta) \|v(\eta)\|_p^p d\eta \\ &\geq c(t+d)^{-\lambda} \ln^{-\nu}(1+t+d) \int_0^t (t-\eta) \eta^p d\eta \\ &= cB(2, 1+p) t^{2+p} (t+d)^{-\lambda} \ln^{-\nu}(1+t+d). \end{aligned}$$

We are now in a position to present the main results.

Theorem 1. *Assume* (H). *When $n = 1$, if $1 < pq < \infty$, every weak solution of (1.1) blows up in finite time.*

Proof. Consider two cases. For simplicity we let $\mu = pq$.

Case 1. $0 < p < 1 < q$. Since $\|u(t)\|_1 \leq c\|u(t)\|_\mu (t+d)^{\frac{\mu-1}{\mu}}$, by (2.14) we have

$$\begin{aligned}\|u(t)\|_\mu \geq{}& (\tilde{c}_1/c)t^{2+p}(t+d)^{-\frac{\mu-1}{\mu}} \\ &+ (\tilde{c}_2/c)(t+d)^{-\frac{3\mu-2p\mu-1}{\mu}} \int_0^t (t-\tau)^3 \|u(\tau)\|_\mu^\mu d\tau.\end{aligned} \tag{2.21}$$

Thus $\|u(t)\|_\mu \geq \psi(t)$, the nonnegative solution of (2.2) with $\alpha = 2+p$, $\gamma = (\mu-1)/\mu$, and $\sigma = (3\mu - 2p\mu - 1)/\mu$. Lemma 1 then implies that u, and hence the solution of (1.1) cannot exist globally.

Case 2. $1 \leq p \leq q$. Since v is also compactly supported, by Hölder's inequality we get

$$\|v(t)\|_1 = \int\limits_{|x|<t+d} |v| dx \leq c\|v(t)\|_p (t+d)^{\frac{p-1}{p}}. \tag{2.22}$$

Then by (2.16), (2.22), and the inequality $a^q + b^q \geq 2^{1-q}(a+b)^q$, we obtain

$$\begin{aligned}\|v(t)\|_p^{\frac{1}{q}} \geq{}& 2^{\frac{1-q}{q}}(\hat{c}_1/c)^{\frac{1}{q}} t^{\frac{1}{q}} (t+d)^{-\frac{p-1}{\mu}} \\ &+ 2^{\frac{1-q}{q}}(\hat{c}_2/c)^{\frac{1}{q}}(t+d)^{-\frac{3\mu-2p-1}{\mu}} \int_0^t (t-\tau)^3 \left(\|v(\tau)\|_p^{\frac{1}{q}}\right)^\mu d\tau.\end{aligned} \tag{2.23}$$

Because $\|v(t)\|_p^{\frac{1}{q}} \geq \psi(t)$ with $\alpha = 1/q$, $\gamma = (p-1)/\mu$, and $\sigma = (3\mu - 2p - 1)/\mu$, the solution of (1.1) must blow up in finite time.

Theorem 2. *Assume* (H). *When $n = 2$, if $1 < pq < \infty$ for $0 < p \leq 2$ or $1 < pq < (5p+2)/(p-2)$ for $2 < p < (3+\sqrt{17})/2$, every weak solution of (1.1) blows up in finite time.*

Proof. Let $\mu = pq$. We consider two cases.

Case 1. $0 < p < 1 < q$. Since $\|u(t)\|_1 \leq c\|u(t)\|_\mu (t+d)^{\frac{2(\mu-1)}{\mu}}$, in view of (2.14) we have

$$\begin{aligned}\|u(t)\|_\mu \geq{}& (\tilde{c}_1/c)t^{2+p}(t+d)^{-\frac{2(\mu-1)}{\mu}} \\ &+ (\tilde{c}_2/c)(t+d)^{-\frac{4\mu-2p\mu-2}{\mu}} \int_0^t (t-\tau)^3 \|u(\tau)\|_\mu^\mu d\tau.\end{aligned} \tag{2.24}$$

Thus by Lemma 1, problem (1.1) has no global solutions.

Case 2. $1 \leq p \leq q$. Since $\|v(t)\|_1 \leq c\|v(t)\|_p (t+d)^{\frac{2(p-1)}{p}}$, we find

$$\begin{aligned}\|v(t)\|_p^{\frac{1}{q}} \geq{}& 2^{\frac{1-q}{q}}(\hat{c}_1/c)^{\frac{1}{q}} t^{\frac{1}{q}} (t+d)^{-\frac{2(p-1)}{\mu}} \\ &+ 2^{\frac{1-q}{q}}(\hat{c}_2/c)^{\frac{1}{q}}(t+d)^{-\frac{4\mu-2p-2}{\mu}} \int_0^t (t-\tau)^3 \left(\|v(\tau)\|_p^{\frac{1}{q}}\right)^\mu d\tau.\end{aligned} \tag{2.25}$$

If $1/q \geq 2(p-1)/\mu$, i.e., $p \leq 2$, since $(4\mu - 2p - 2)/\mu < 4$, there is no any upper bound on pq. For $p > 2$, let $1/\mu < \theta < 1/q$. By (2.17), (2.19), and the inverse Hölder inequality, we have that

$$\begin{aligned}\|u(t)\|_q^q &\geq c^{-q}\|u(t)\|_1^q (t+d)^{-2(q-1)}\\ &\geq (c_0^{1-\theta}/c)^q \|u(t)\|_1^{\theta q} t^{q(1-\theta)(2+p)}(t+d)^{-\lambda q(1-\theta)-2(q-1)} \ln^{-\nu q(1-\theta)}(1+t+d)\\ &\geq (c_0^{1-\theta}/c)^q t^{q(1-\theta)(2+p)}(t+d)^{-\lambda q(1-\theta)-2(q-1)}\\ &\quad \cdot \ln^{-\nu q(1-\theta)}(1+t+d)\left[\int_0^t (t-\tau)\|v(\tau)\|_p^p d\tau\right]^{\theta q}\\ &\geq (c_0^{1-\theta}/c)^q \left(\frac{t}{t+d}\right)^{\beta}(t+d)^{-\tilde{\lambda}}\ln^{-\kappa}(1+t+d)\int_0^t (t-\tau)\|v(\tau)\|_p^{\theta\mu} d\tau,\end{aligned}$$

where $\beta = q(1-\theta)(2+p)$, $\tilde{\lambda} = \lambda q(1-\theta) + 2(q-1) + 2(1-\theta q) - \beta$, and $\kappa = \nu q(1-\theta)$. Clearly, $\tilde{\lambda} > 0$. Thus we find that

$$\begin{aligned}\|v(t)\|_p &\geq (c_k/c)t(t+d)^{-\frac{2(p-1)}{p}} + c^{-1}(t+d)^{-\frac{2(p-1)}{p}}\int_0^t (t-\eta)\|u(\eta)\|_q^q d\eta\\ &\geq (c_k/c)t(t+d)^{-\frac{2(p-1)}{p}} + \left(c_0^{1-\theta}/c^{\frac{q+1}{q}}\right)^q (t+d)^{-\frac{2(p-1)}{p}}\\ &\quad \cdot \int_0^t \left(\frac{\eta}{\eta+d}\right)^{\beta}(\eta+d)^{-\tilde{\lambda}}\ln^{-\kappa}(1+\eta+d)(t-\eta)\int_0^{\eta}(\eta-\tau)\|v(\tau)\|_p^{\theta\mu} d\tau d\eta\\ &\geq (c_k/c)t(t+d)^{-\frac{2(p-1)}{p}} + \left(c_0^{1-\theta}/c^{\frac{q+1}{q}}\right)^q (t+d)^{-\frac{2(p-1)}{p}-\tilde{\lambda}}\ln^{-\kappa}(1+t+d)\\ &\quad \cdot \int_0^t \|v(\tau)\|_p^{\theta\mu}\int_{\tau}^t \left(\frac{\eta}{\eta+d}\right)^{\beta}(t-\eta)(\eta-\tau) d\eta d\tau\\ &\geq (c_k/c)t(t+d)^{-\frac{2(p-1)}{p}} + \left(c_0^{1-\theta}/c^{\frac{q+1}{q}}\right)^q B(2,2)(t+d)^{-\frac{2(p-1)}{p}-\tilde{\lambda}}\\ &\quad \cdot \ln^{-\kappa}(1+t+d)\int_0^t (t-\tau)^3\left(\frac{\tau}{\tau+d}\right)^{\beta}\|v(\tau)\|_p^{\theta\mu} d\tau.\end{aligned}$$

Hence $\|v(t)\|_p \geq \psi(t)$, the nonnegative solution of (2.11) with $\gamma - \alpha = (p-2)/p$ and $\sigma = 2(p-1)/p + \tilde{\lambda}$. Conditions in Lemma 2 then require

$$\frac{p-2}{p} < \frac{4p - 2(p-1) - \tilde{\lambda}p}{p(\theta\mu - 1)}, \tag{2.26}$$

or

$$(p-2)(\theta\mu - 1) < 2p + 2 - \frac{1}{2}p\mu + \frac{1}{2}p\theta\mu + \mu - \theta\mu. \tag{2.27}$$

Solving this inequality yields

$$\left(\frac{1}{2}p - 1\right)(\theta + 1)\mu < 3p. \tag{2.28}$$

Since $\theta\mu$ can be chosen arbitrarily close to one, (2.28) leads to the limitation

$$pq < \frac{5p+2}{p-2}. \tag{2.29}$$

Moreover, setting $q = p$ in (2.29) we obtain

$$p^3 - 2p^2 - 5p - 2 < 0, \tag{2.30}$$

that is,

$$(p+1)(p^2 - 3p - 2) < 0, \tag{2.31}$$

which is satisfied if $p < (3 + \sqrt{17})/2$.

Remark 3. Such a result contrasts sharply with that of (1.4), wherein pq is uniformly bounded from above by $(1 + 2/n)^2$. This observation indicates that system (1.1) is a more unstable problem than (1.4).

Theorem 3. *Assume* (H). *When* $n = 3$, *if* $1 < pq < (p+3)/(1-p)$ *for* $0 < p < 1$ *or* $1 < pq < (3p+1)/(p-1)$ *for* $1 \le p < 1+\sqrt{2}$, *every weak solution of (1.1) blows up in finite time.*

Proof. Set $\mu = pq$. We again consider two cases.

Case 1. $0 < p < 1 < q$. Making use of (2.14) and the estimate $\|u(t)\|_1 \le c\|u(t)\|_\mu (t+d)^{\frac{3(\mu-1)}{\mu}}$, we have

$$\begin{aligned} \|u(t)\|_\mu \ge {} & (\tilde{c}_1/c) t^{2+p} (t+d)^{-\frac{3(\mu-1)}{\mu}} \\ & + (\tilde{c}_2/c)(t+d)^{-\frac{5\mu-2p\mu-3}{\mu}} \int_0^t (t-\tau)^3 \|u(\tau)\|_\mu^\mu d\tau. \end{aligned} \tag{2.32}$$

If $2 + p \ge 3(\mu-1)/\mu$, i.e., $\mu \le 3/(1-p)$, $(5\mu - 2p\mu - 3)/\mu < 4$ is satisfied. Then Lemma 1 ensures the finite time blow–up of the solution of (1.1). If $\mu > 3/(1-p)$, from the restriction $\gamma - \alpha < (4-\sigma)/(\mu-1)$ in Lemma 2, it follows that

$$\frac{\mu(1-p) - 3}{\mu} < \frac{2p\mu - \mu + 3}{\mu(\mu-1)}, \tag{2.33}$$

which implies $pq < (p+3)/(1-p)$.

Case 2. $1 \le p \le q$. For $p = 1$, since $\|u(t)\|_1 \ge \int_0^t (t-\eta)\|v(\eta)\|_1 d\eta$ and $\|v(t)\|_1 \ge c_k t + \int_0^t (t-\tau)\|u(\tau)\|_q^q d\tau$,

$$\begin{aligned} \|u(t)\|_q \ge {} & (c_k/c) B(2,2) t^3 (t+d)^{-\frac{3(q-1)}{q}} \\ & + c^{-1} B(2,2)(t+d)^{-\frac{3(q-1)}{q}} \int_0^t (t-\tau)^3 \|u(\tau)\|_q^q d\tau, \end{aligned}$$

which upon application of Lemma 1, yields $1 < q < \infty$.

For $p > 1$, let $1/\mu < \theta < 1/q$. By (2.19) and the estimate $\|u(t)\|_1 \le c\|u(t)\|_q (t+d)^{\frac{3(q-1)}{q}}$, and applying the inverse Hölder inequality, one can see that

$$\begin{aligned} \|u(t)\|_q^q & \ge (c_0^{1-\theta}/c)^q \|u(t)\|_1^{\theta q} t^{q(1-\theta)(2+p)} (t+d)^{-\lambda q(1-\theta) - 3(q-1)} \\ & \ge (c_0^{1-\theta}/c)^q \left(\frac{t}{t+d}\right)^\beta (t+d)^{-\hat{\lambda}} \int_0^t (t-\tau)\|v(\tau)\|_p^{\theta\mu} d\tau, \end{aligned} \tag{2.34}$$

where $\beta = q(1-\theta)(2+p)$, $\hat{\lambda} = \lambda q(1-\theta) + 3(q-1) + 2(1-\theta q) - \beta$. Since $\|v(t)\|_1 \leq c\|v(t)\|_p (t+d)^{\frac{3(p-1)}{p}}$, it follows that

$$\begin{aligned} \|v(t)\|_p &\geq (c_k/c)t(t+d)^{-\frac{3(p-1)}{p}} + c^{-1}(t+d)^{-\frac{3(p-1)}{p}} \int_0^t (t-\eta)\|u(\eta)\|_q^q d\eta \\ &\geq (c_k/c)t(t+d)^{-\frac{3(p-1)}{p}} + \left(c_0^{1-\theta}/c^{\frac{q+1}{q}}\right)^q B(2,2)(t+d)^{-\frac{3(p-1)}{p}-\hat{\lambda}} \\ &\quad \cdot \int_0^t (t-\tau)^3 \left(\frac{\tau}{\tau+d}\right)^\beta \|v(\tau)\|_p^{\theta\mu} d\tau, \end{aligned} \tag{2.35}$$

provided $\hat{\lambda} > 0$, which is equivalent to the following inequality

$$1 < (p-1)q + (2-p)\theta q. \tag{2.36}$$

Note that such a requirement is superfluous for the case $1 < p \leq 3/2$, since if $\hat{\lambda} \leq 0$, one can choose a different β $(< q(1-\theta)(2+p))$ in (2.34) such that the corresponding $\hat{\lambda}$ satisfies $0 < \hat{\lambda} < 4 - 3(p-1)/p$. Consequently,

$$\begin{aligned} \|v(t)\|_p &\geq (c_k/c)t(t+d)^{-\frac{3(p-1)}{p}} + \left(c_0^{1-\theta}/c^{\frac{q+1}{q}}\right)^q B(2,2)(t+d)^{-\frac{3(p-1)}{p}-\hat{\lambda}} \\ &\quad \cdot \int_0^t (t-\tau)^3 \left(\frac{\tau}{\tau+d}\right)^\beta \tau^{q(1-\theta)(2+p)-\beta} \|v(\tau)\|_p^{\theta\mu} d\tau, \end{aligned}$$

then based on the argument used to prove Lemma 1, it can be shown that $\|v(t)\|_p$ fails to exist globally for every $pq > 1$. Thus we only need to discuss the validity of (2.36) for the case $p > 3/2$. Consider two possibilities.

(i) $3/2 < p \leq 2$. In view of the lower bound on θ, (2.36) is valid if $1 < (p-1)q + (2-p)/p$, which is satisfied for $3/2 < p \leq q$.

(ii) $p > 2$. By virtue of the upper bound on θ, (2.36) holds if $1 < (p-1)q + (2-p)$, which is true for any $q \geq p$.

Therefore, in what follows, we may always assume $\hat{\lambda} > 0$ and focus our attention on the inequality (2.35). If $p \leq 3/2$, the condition $0 < \sigma < 4$ in Lemma 1 restricts $(p-1)\mu + (2-p)\theta\mu < 2p + 3$, which implies $pq < (3p+1)/(p-1)$. For $p > 3/2$, Lemma 2 will hold if one requires

$$(2p-3)(\theta\mu - 1) < 2p + 3 - p\mu + p\theta\mu + \mu - 2\theta\mu, \tag{2.37}$$

or

$$(p-1)(\theta+1)\mu < 4p, \tag{2.38}$$

which yields the bound

$$pq < \frac{3p+1}{p-1}. \tag{2.39}$$

Furthermore, for $q = p$, (2.39) becomes

$$(p+1)(p^2 - 2p - 1) < 0,$$

which can be valid only if $p < 1 + \sqrt{2}$. Thus the proof is completed.

Remark 4. It is worth noting that results from the above theorems show that problem (1.1) for $n = 1$ is the most unstable while it is the least unstable for $n = 3$. Such properties are well consistent with those possessed by the solutions u_0, v_0 of $w_{tt} = \Delta w$, since when $n = 1$ u_0, v_0 do not decay uniformly to zero as $t \to \infty$ while Huygens' Principle holds when $n = 3$.

Acknowledgments

The author would like to thank Professor P. E. Sacks for informative discussions. He would also like to thank the referee for the helpful comments.

References

1. M. Escobedo and M.A. Herrero, *Boundedness and blow up for a semilinear reaction–diffusion system*, J. Differential Equations **89** (1991), 176–202. MR **91j:**35040
2. H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **16** (1966), 105–113. MR **35:**5761
3. R.T. Glassey, *Finite–time blow–up for solutions of nonlinear wave equations*, Math. Z. **177** (1981), 323–340. MR **82i:**35120
4. R.T. Glassey, *Existence in the large for* $\Box u = F(u)$ *in two space dimensions*, Math. Z. **178** (1981), 233–261. MR **84h:**35106
5. J.W. Jerome, *Approximation of nonlinear evolution systems*, Math. in Sci. Engineering **164**, Academic Press, New York, 1983. MR **85g:**35064
6. F. John, *Blow–up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), 235–268. MR **80i:**35114
7. T. Kato, *Blow–up of solutions of some nonlinear hyperbolic equations*, Comm. Pure Appl. Math. **32** (1980), 501–505. MR **82f:**35128
8. S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Lectures in Appl. Math. **23** (1986), 293–326. MR **87h:**35217
9. H.A. Levine, *Instability and nonexistence of global solutions of nonlinear wave equation of the form* $Pu_{tt} = -Au + \mathcal{F}(u)$, Trans. Amer. Math. Soc. **192** (1974), 1–21. MR **49:**9436
10. H.A. Levine, *A Fujita type global existence–global nonexistence theorem for a weakly coupled system of reaction–diffusion equations*, Z. Angew Math. Phys. **42** (1991), 408–430. MR **92g:**35097
11. J. Schaeffer, *The equation* $u_{tt} - \Delta u = |u|^p$ *for the critical value* p, Proc. Roy. Soc. Edinburgh Sect. A **101** (1985), 31–44. MR **87g:**35159
12. T. Sideris, Ph.D. thesis, Indiana University, Bloomington, 1981.
13. T. Sideris, *Nonexistence of global solutions of semilinear wave equations in high dimensions*, J. Differential Equations **52** (1984), 378–406. MR **86d:**35090
14. F.B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math. **38** (1981), 29–40. MR **82g:**35059

Department of Mathematics, University of Southwestern Louisiana, Lafayette, Louisiana 70504

E-mail address: kxd5858@usl.edu